Topics in Mathematical Imaging
Lecture 1

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Lecture plan

- Lecture 1: Variational models & PDEs for imaging by examples
- Lecture 2: Derivation of these models & analysis
- Lecture 3: Numerical solution
- Lecture 4: Some machine learning connections

The variational approach

General task: **restore** $\mathbf{u}$ from an **observed datum** $\mathbf{g}$ where

$$g = \underbrace{Tu}_{\text{forward model}} + \underbrace{n}_{\text{noise}}.$$

Variational approach: Compute u as a minimizer of

$$\mathcal{J}(u) = \alpha \underbrace{R(u)}_{\text{regularization}} + \underbrace{D(Tu, g)}_{\text{data fidelity}} \to \min_{u \in B},$$

where

- $R(u)$ is a prior/regularizer that models a-priori information on u weighted by positive α, e.g., $R(u) = \|\nabla u\|_{L^1}$
- $D(\cdot, \cdot)$ is a distance function, e.g. $D(Tu, g) = \|Tu - g\|_2^2$ and B suitable Banach space, e.g., $B = BV(\Omega)$.

Engl, Hanke, Neubauer '96; Natterer, Wübbeling '01; Kaltenbacher, Neubauer, Scherzer '08; Schuster, Kaltenbacher, Hofmann, Kazimierski '12

Sparsity promoting regularisation



Undersampled Fourier

MRI: measured datum is a sampled Fourier transform; nonadaptive compressed acquisition, i.e., compressed sensing

$$g = (\mathcal{F}u)_{|\Lambda} + n$$



Undersampled Fourier

Goal: identify a piecewise constant function u consistent to the datum

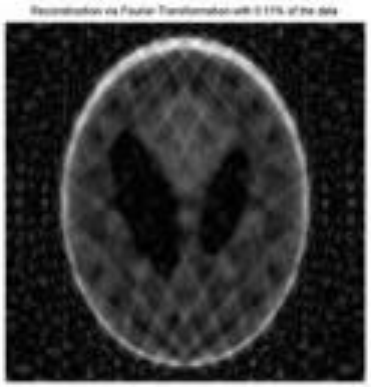
Sparsity in $\nabla u \Rightarrow$ NP-hard problem!

Convex relaxation $\Rightarrow$ ℓ_1- and total variation minimization

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|(\mathcal{F}u)_{|\Lambda} - g\|^2 \to \min_u,$$

Rudin, Osher, Fatemi, Physica D '92; Candes, Romberg, Tao, IEEE Trans Inf Theory '06

Sparsity promoting regularisation



Zero-filling solution



Undersampled Fourier

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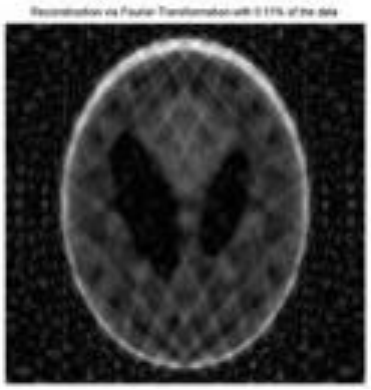
Shepp-Logan Phantom



Sparsity promoting regularisation



Zero-filling solution



Total variation solution

MRI: measured datum is a sampled Fourier transform; nonadaptive compressed acquisition, i.e., compressed sensing

$$g = (\mathcal{F}u)_{|\Lambda} + n$$

Goal: identify a piecewise constant function u consistent to the datum

Sparsity in $\nabla u \Rightarrow$ NP-hard problem!

Convex relaxation $\Rightarrow \ell_1$- and total variation (TV) minimization

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|(\mathcal{F}u)_{|\Lambda} - g\|^2 \to \min_u,$$

Rudin, Osher, Fatemi, Physica D '92; Candes, Romberg, Tao, IEEE Trans Inf Theory '06

Connection to nonlinear PDEs

Nonlinear image smoothing with total variation regularisation

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|u-g\|^2 \to \min_u$$

with steepest descent

$$u_t = \alpha\, p + (u-g), \quad p \in \partial\|\nabla u\|_1, \quad \text{in } \Omega,$$

Perona, Malik '90; Rudin, Osher, Fatemi, Physica D '92; and various contributions from Ambrosio, Caselles, Chambolle, Lions, Morel, Novaga, ...

Connection to nonlinear PDEs

Nonlinear image smoothing with total variation regularisation

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|u-g\|^2 \to \min_u$$

with steepest descent for $|\nabla u| \neq 0$

$$u_t = \alpha \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + (u-g), \quad \text{in } \Omega,$$

Perona, Malik '90; Rudin, Osher, Fatemi, Physica D '92; and various contributions from Ambrosio, Caselles, Chambolle, Lions, Morel, Novaga, ...

Modelling idea . . .

Taking advantage of the property of solutions of such PDEs and variational problems to be characterised by a few relevant features with certain geometric properties, e.g. curvature, which are recovered by adaptive nonlinear iterations.

4D reconstruction from sub-sampled data

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4D Bregman-TV Reconstruction from 20% line-sampling

$$\text{Acquire } \mathcal{SF}(u) \Rightarrow \text{ reconstruct } u(x,t) = \text{argmin}_v \alpha \, \|\nabla v\|_1 + \|\mathcal{SF}(v) - g\|_2^2$$

M. Benning, A. Sederman, CBS, L. Gladden, et al.; M. Benning, L. Gladden, D. Holland, CBS, and T. Valkonen, J. Magnetic Resonance 238, pp. 26 - 43, 2014; Osher, Burger et al. '05; M. Lustig et al. '07

4D reconstruction from sub-sampled data

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Fourier inversion from 20% line-sampling

$$\text{Acquire } \mathcal{S}\mathcal{F}(u) \Rightarrow \text{ reconstruct } u(x,t) \text{ by zero-filling.}$$

M. Benning, A. Sederman, CBS, L. Gladden, et al.

4D reconstruction from sub-sampled data

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4D Bregman-TV Reconstruction from 20% line-sampling

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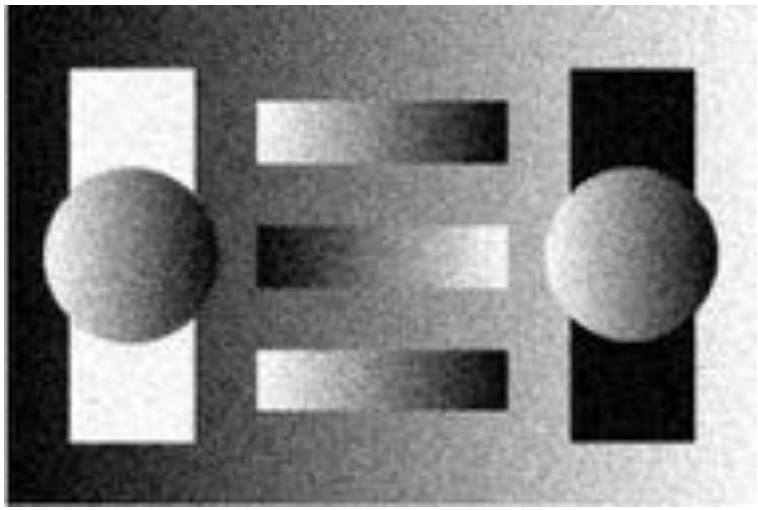
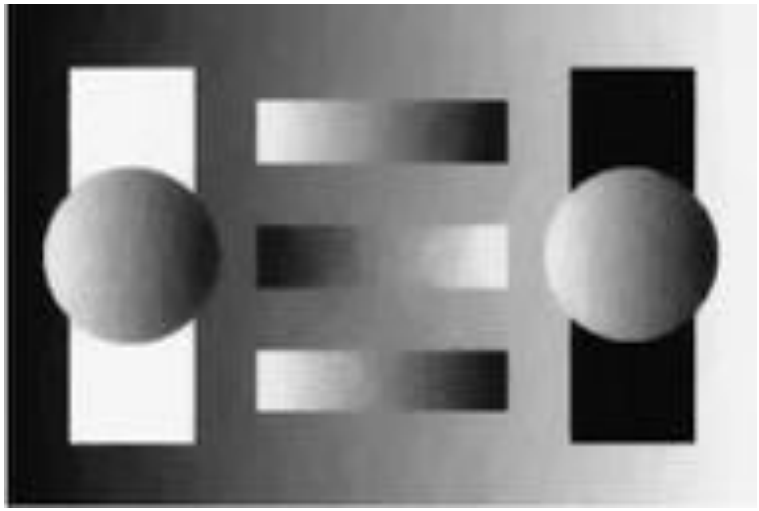
What is the right sparsity?

$\min_u \{\alpha\|\nabla u\|_1 + \|u - f\|_2^2\}$



Noisy image



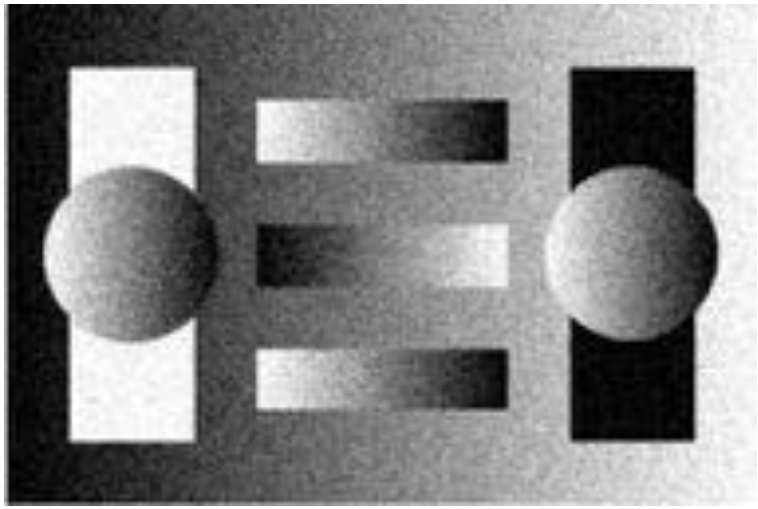
TV denoised image

Image courtesy of K. Papafitsoros

References: Rudin, Osher, Fatemi '92; Hinterberger, Scherzer, Computing '06; Bredies, Kunisch, Pock, SIAM Imaging '10; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

What is the right sparsity?

$\min_u \{\alpha\|\nabla^2 u\|_1 + \|u - f\|_2^2\}$



Noisy image



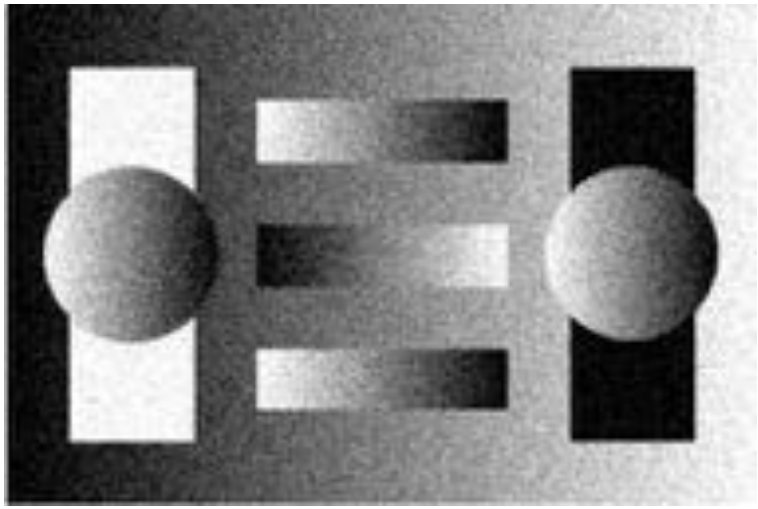
TV^2 denoised image

Image courtesy of K. Papafitsoros

References: Rudin, Osher, Fatemi '92; Hinterberger, Scherzer, Computing '06; Bredies, Kunisch, Pock, SIAM Imaging '10; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

What is the right sparsity?

$$\min_u \left\{ \min_w \{\alpha_1 \|\nabla u - w\|_1 + \alpha_2 \|Ew\|\} + \|u - f\|_2^2 \right\}$$



Noisy image



TGV^2 denoised image

Image courtesy of K. Papafitsoros

References: Rudin, Osher, Fatemi '92; Hinterberger, Scherzer, Computing '06; Bredies, Kunisch, Pock, SIAM Imaging '10; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 ...

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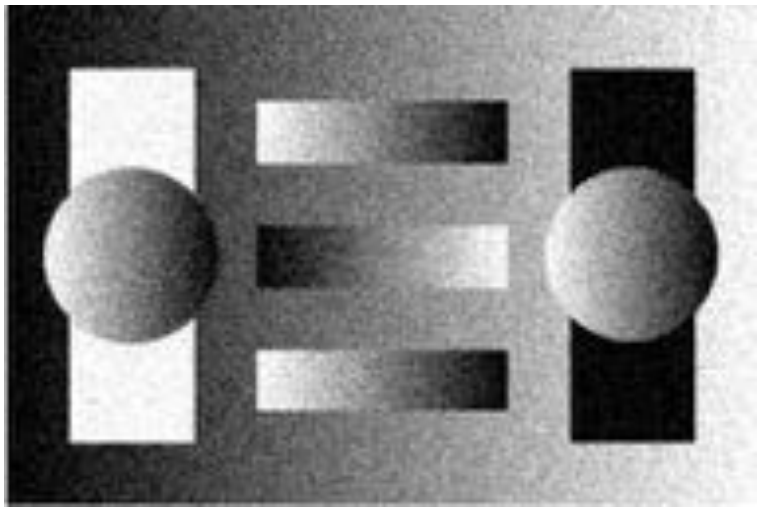

Noisy image



TGV^2 denoised image

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Disclaimer: there many more ways to enforce sparsity: wavelets, curvelets, shearlets, learned dictionaries, . . . that I will not discuss

Is sparsity enough?

$$J(u, \mathbf{v}) = \int_0^T \frac{1}{2} \|Ku - f\|_2^2 \, dt + \alpha \int_0^T R(u(\cdot, t))^p dt + \beta \int_0^T S(\mathbf{v}(\cdot, t))^q dt$$

$$\text{s.t.} \quad u_t + \nabla \cdot (u\mathbf{v}) = 0 \quad \text{resp.} \quad u_t + \nabla u \cdot \mathbf{v} = 0$$

$$\|\boldsymbol{v}\|_\infty \leq c_\infty, \|\nabla \cdot \boldsymbol{v}\|_\Theta \leq c^*$$

with $1 < p, q \in \mathbb{R}$, K bounded and linear acting on appropriate function space.

B. Horn, B. Schunk, AI '81; J. D. Benamou, J. Brenier, Numerische Mathematik '00; M. Burger, H. Dirks, CBS, SIIMS '17; M. Burger, M. Drechsler, L. Lang, I. Palacios, CBS '18

Dynamic modelling

Joint work with Martin Burger, Hendrik Dirks, Lukas Lang, Jocelyn Etienne and Guy Blanchard







Dynamic modelling

Optical Flow

Assumption: **constant brightness**

$$u(x,t) = u(x + \delta_t \mathbf{v}(x,t), t + \delta_t)$$

Leads to:
optical flow equation:

$$\mathbf{v} \cdot \nabla u + u_t = 0.$$

Mass preservation

Assumption: **continuous flow, no teleportation.**

$$\frac{d}{dt} \int_\Omega u(x,t) = 0$$

Leads to:
continuity equation:

$$\nabla \cdot (\mathbf{v} u) + u_t = 0.$$

Challenge: underdetermined problem $\Rightarrow$ regularisation.

Dynamically informed reconstruction

Figure: 50% destroyed

Figure: Restoration

M. Burger, H. Dirks, CBS, arXiv:1607.03255; L. Lang, J. Etienne, G. Blanchard, CBS, in preparation; A. Hauptmann et al. '17; F. Lucka et al. '17

Reconstruction without dynamics

Figure: 50% destroyed

Figure: Restoration

M. Burger, H. Dirks, CBS, arXiv:1607.03255; L. Lang, J. Etienne, G. Blanchard, CBS, in preparation; A. Hauptmann et al. '17; F. Lucka et al. '17

Multi-Contrast MRI Reconstruction

Work of Simon Arridge, Matthias Ehrhardt and Marta Betcke





Idea: make use of correlation in multi-modal imaging and reconstruct jointly

Multi-Contrast MRI Reconstruction

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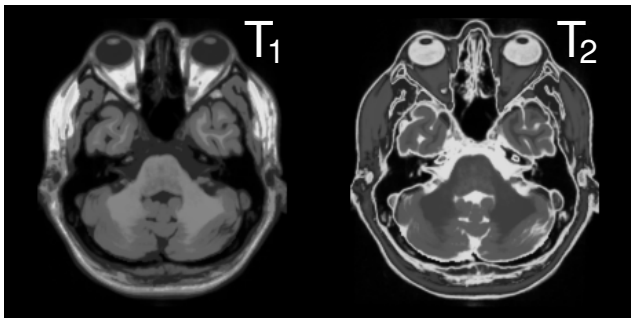
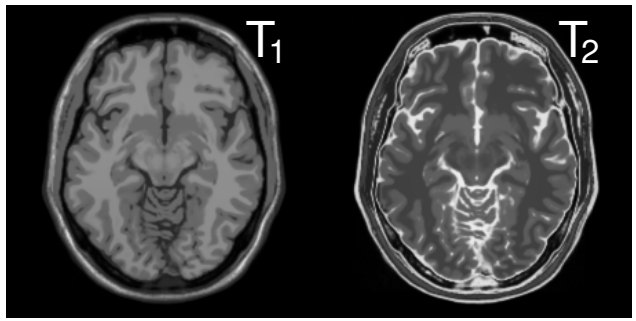
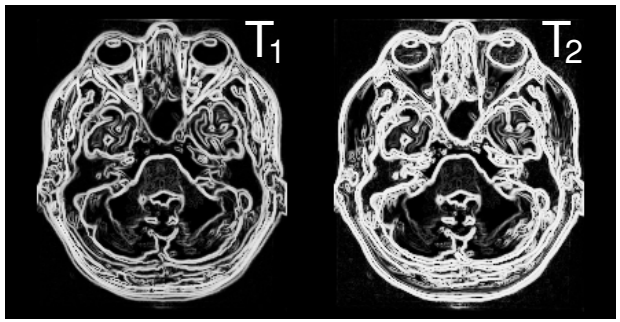
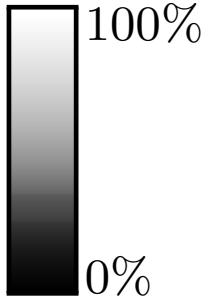
- versatile technology with many **different contrasts**
- MRI contrasts show **similar structures** due to same anatomy
- **exploit redundancy** and reconstruct with less data
- **shorter scan**: patient comfort, save time, dynamic imaging



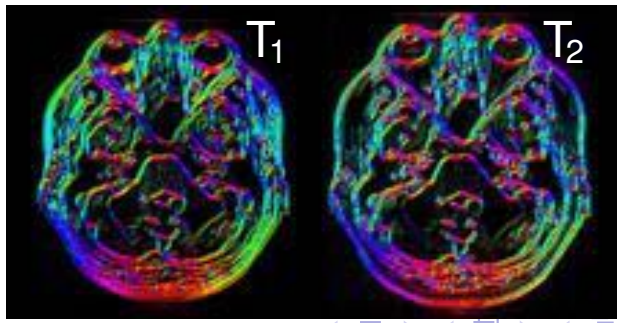


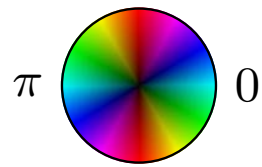
- Difficult to compare images of different contrasts
- Base image structure on location or direction of gradients





direction





Multi-Contrast MRI Reconstruction

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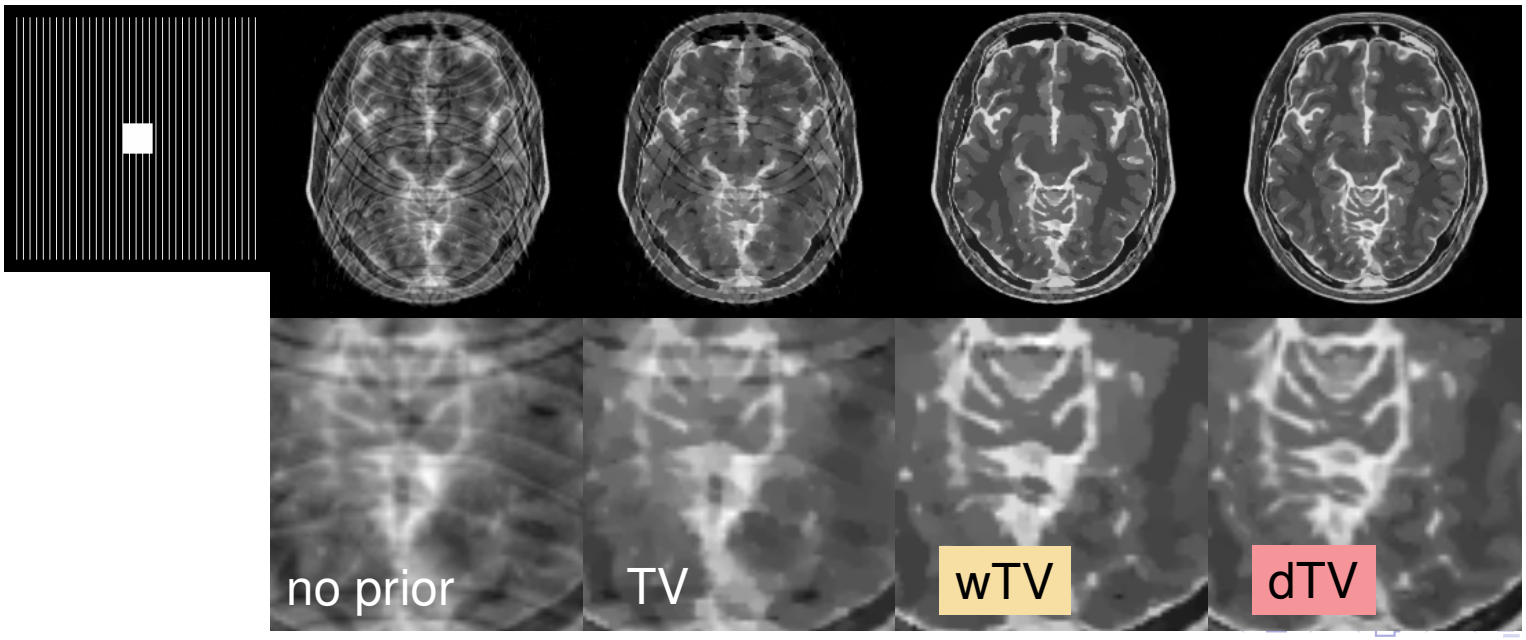
$$\operatorname*{argmin}_u \left\{ \frac{1}{2}|Au-b|^2 + \alpha \int_\Omega |\mathcal{D}(x)\nabla u(x)|dx \right\}$$

Structure either based on location [Arridge et al., 2007]:

$$\mathcal{D}(x) = \eta/|\nabla v(x)|_\eta, \quad |\nabla v(x)|_\eta = \sqrt{|\nabla v(x)|^2 + \eta^2}, \qquad (\text{wTV})$$

or direction [Ehrhardt and Betcke, 2015, arxiv]:

$$\mathcal{D}(x) = I - \xi(x)\xi^T(x), \quad \xi(x) = \nabla v(x)/|\nabla v(x)|_\eta. \qquad (\text{dTV})$$



Computed tomography

Principle:

CT demo

Illustration is courtesy of Samuli Siltanen (Helsinki)

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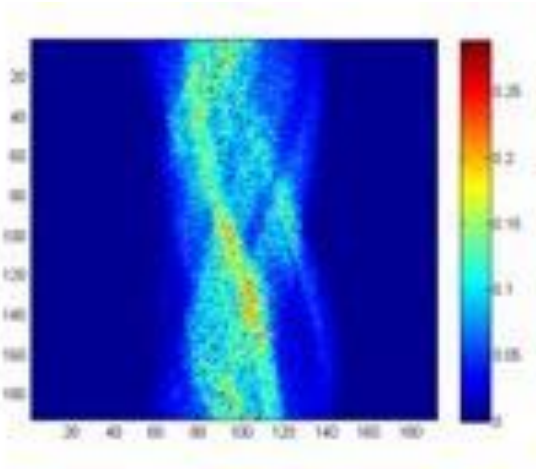
Positron Emission Tomography (reconstruction from Radon samples)



Measurements $f = K(\mathcal{R}u)$, where $\mathcal{R}$ Radon transform and K models imperfection in measurements.

Reconstruct heart of a mouse

$$\alpha_1 \|Du\|_{2,1} + \alpha_2 \|DRu\|_{2,1} + \frac{1}{2}\|(\mathcal{R}u)_{|\Lambda} - f\|^2 \to \min_u$$

References: Barbano, Fokas, CBS, SAMPTA Proceedings '11; Burger, Müller, Papoutsellis, CBS, Inverse Problems '14

Mathematical imaging by examples

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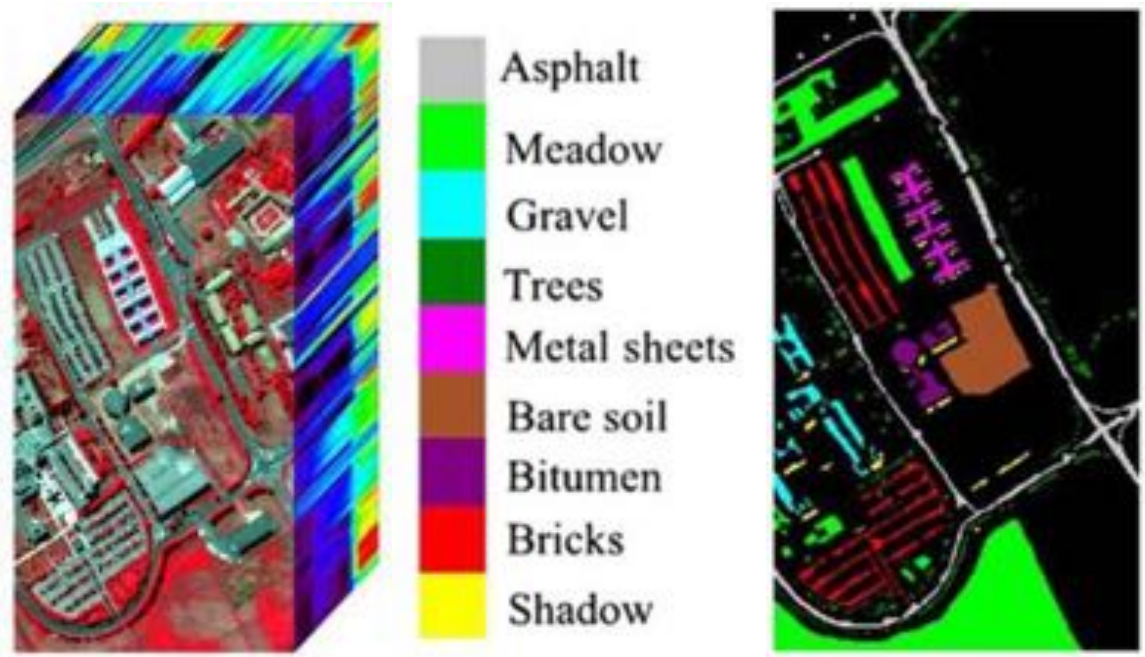
- Image segmentation

- Image inpainting

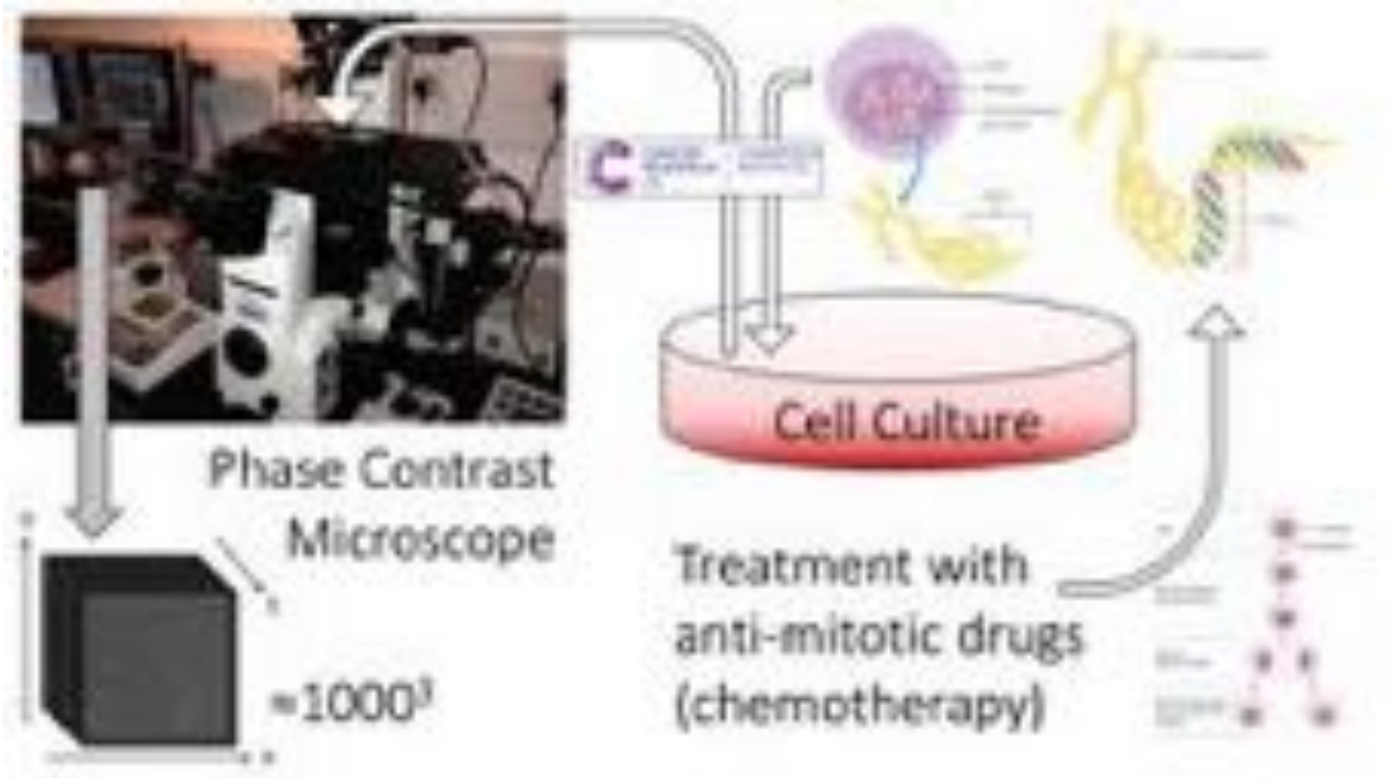
Image segmentation / classification

Segmentation / classification: detect and contour objects of interest and assign them a label (class)



. . . lets look at some examples.

Mitosis Analysis for Cancer Research



Aim: Development of mathematical imaging tools for automated analysis of the acquired image sequences

→ **Mitosis duration** and distribution of **cell fates**

[J. Grah, M. Burger, S. Reichelt, C.-B. Schönlieb et al. MitosisAnalyser 2016]

Mitosis Analysis for Cancer Research

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Joint work with Joana Grah, Martin Burger and Stefanie Reichelt





Automated Mitosis Analysis

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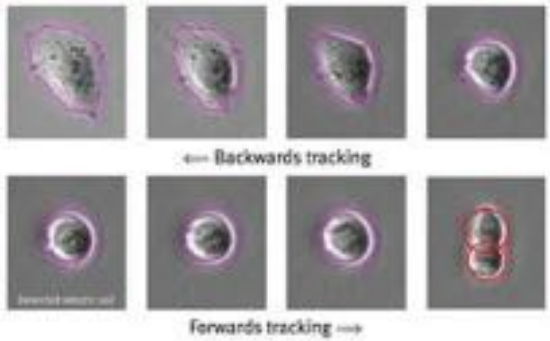
Step 1: Mitosis Detection

Step 2 & 3: Cell Tracking & Cell Fate



Overall: MitosisAnalyser



Example: Multi-modal experiment (**F**luorescent **U**biquitination-based **C**ell **C**ycle **I**ndicator)

	Control			Treatment 1			Treatment 2		
	1	2	3	4	5	6	7	8	9
Number of events	27	21	43	29	21	32	34	15	35
Avg mitosis duration (min)	101	86	80	77	107	106	110	53	111
Total avg mitosis duration	89			97			91		

[J. Grah, S. B. Koo, M. Burger, S. Reichelt, CBS et al. 2016]

Step 2: cell tracking

Minimise energy:

$$E(\phi) = \underbrace{\lambda_1 \int_\Omega \left(1 - H\left(\phi\right)\right) \left(c_{\text{in}} - |v|\right)^2 dx + \lambda_2 \int_\Omega \left(H\left(\phi\right)\right) \left(c_{\text{out}} - |v|\right)^2 dx}_{\text{partition into two regions with different normal velocities}}$$

$$+ \underbrace{\lambda_3 \int_\Omega g_{\text{grad}}\left(|\nabla f_\sigma(x)|\right)\ \delta(\phi(x))\ |\nabla\phi(x)|\ dx}_{\text{stop contour at edges based on the gradient}}$$

$$+ \underbrace{\lambda_4 \int_\Omega g_{\sigma_{\text{loc}}}\left(f_\sigma(x)\right)\ \delta(\phi(x))\ |\nabla\phi(x)|\ dx}_{\text{stop contour at edges based on the local std}}$$

$$+ \underbrace{\mu \int_\Omega \delta\left(\phi(x)\right) |\nabla\phi(x)|\, dx}_{\text{small contour length}}$$

[Grah 14'; Moeller '13; Chan, Vese 98']

Automated Mitosis Analysis

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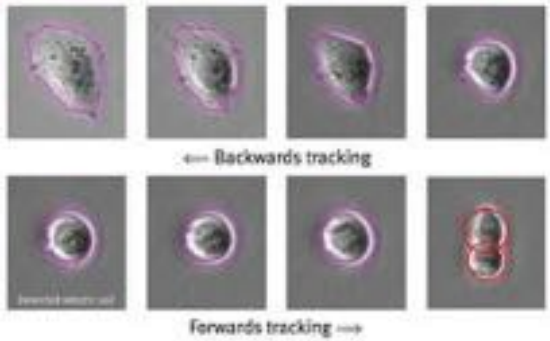
Step 1: Mitosis Detection

Step 2 & 3: Cell Tracking & Cell Fate



Overall: MitosisAnalyser



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[J. Grah, S. B. Koo, M. Burger, S. Reichelt, CBS et al. 2016]

Automated Mitosis Analysis

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Overall: Multi-modal experiment

[J. Grah, S. B. Koo, M. Burger, S. Reichelt, CBS et al. 2016]

MitosisAnalyser

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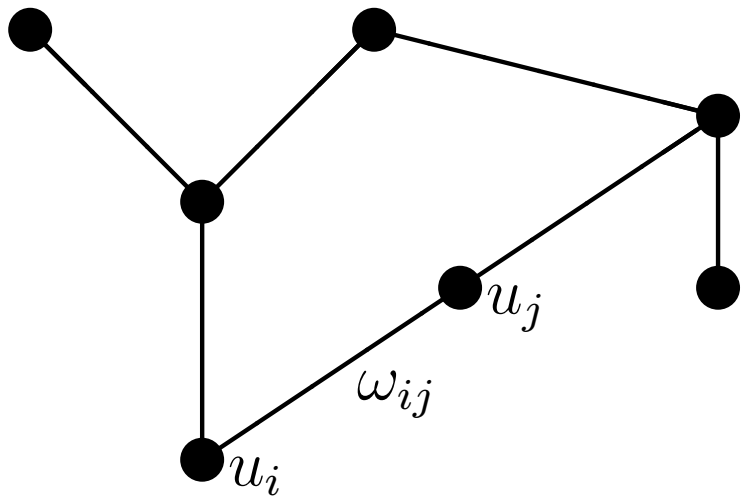
[J. Grah, M. Burger, S. Reichelt, CBS et al., Methods, 115, 2017] **Software:**
`github.com/JoanaGrah/MitosisAnalyser` forward

Change the setting: graphs

Data points (pixels) are represented by nodes in an undirected graph $G = (V, E)$. Similarity is encoded in edge weights ω_{ij}. 'Object' and 'background' are distinguished by node labels $u_i : V \to \mathbb{R}$.



Cluster into two groups: $\{u = 0\}$ and $\{u = 1\}$.

Graph segmentation



(Calatroni, vG, Rowland, Schönlieb, Flenner; 2016)

- Vertices: the pixels
- Edge weight: $\omega_{ij} = e^{-\|x_i - x_j\|^2/\tau}$ where τ is a scale parameter and
- x_i is the feature vector of pixel i
- Constraint: fidelity to data in original image

Tree segmentation for forest conservation

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- Analyse how and why the world's forests are changing
- Use insights gained from such research to inform conservation policy



Tree segmentation for forest conservation

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Joint work with Juheon Lee, Xiaohao Cai and David Coomes







Tree segmentation for forest conservation

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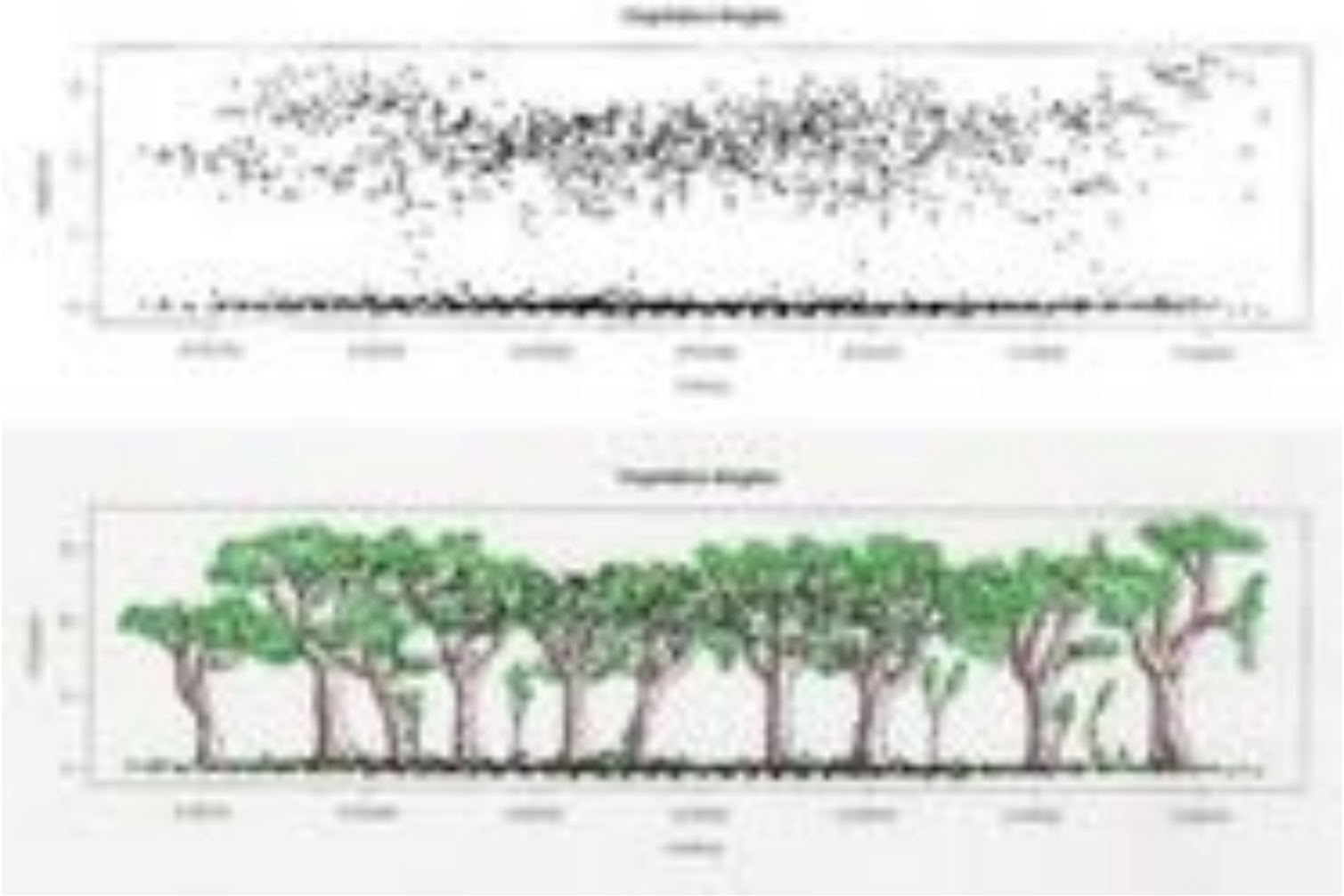
Multi-sensor data: LiDAR



J. Lee, CBS, D. Coomes, et al. '15– Illustration courtesy of Simonson

Tree segmentation for forest conservation

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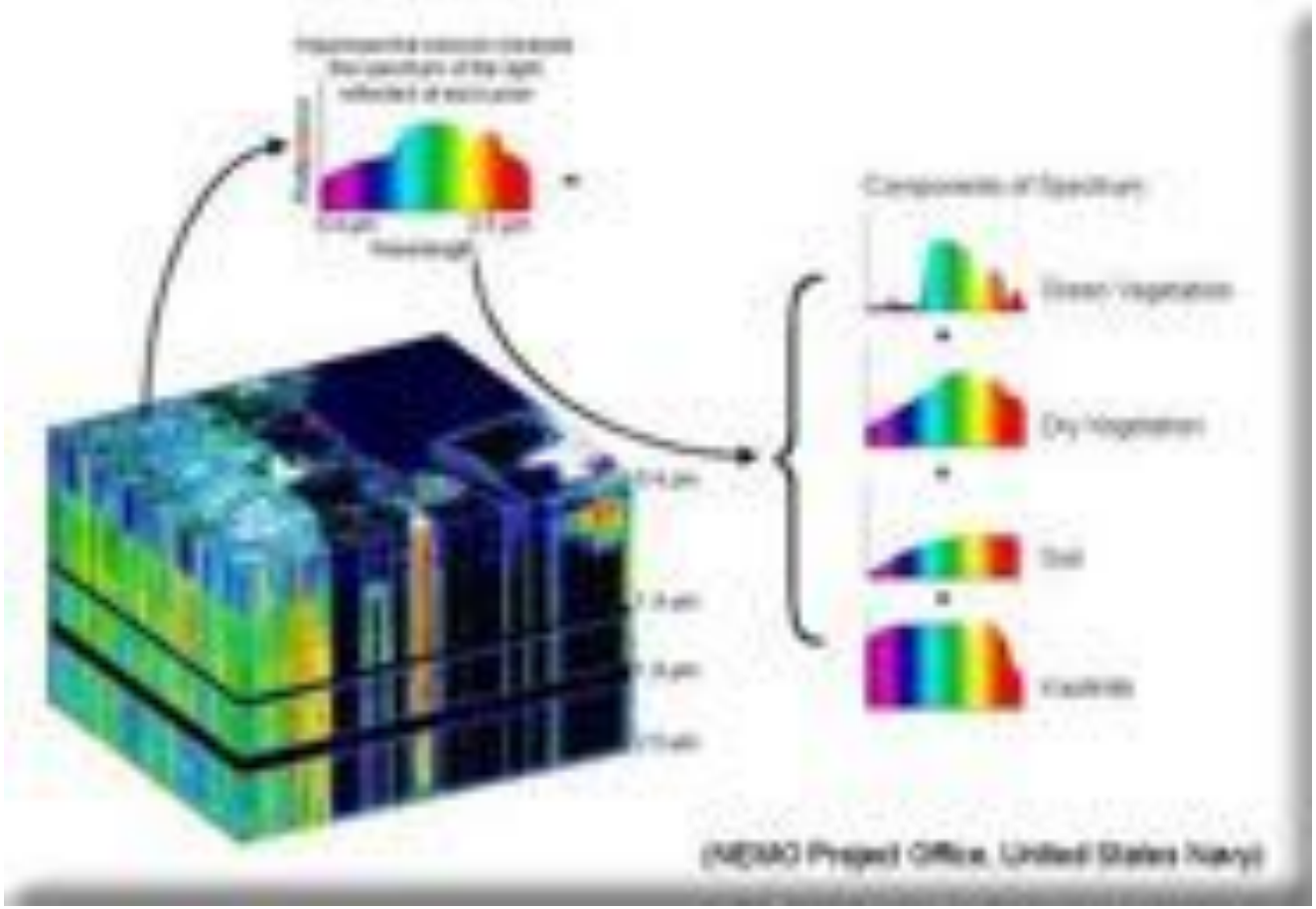
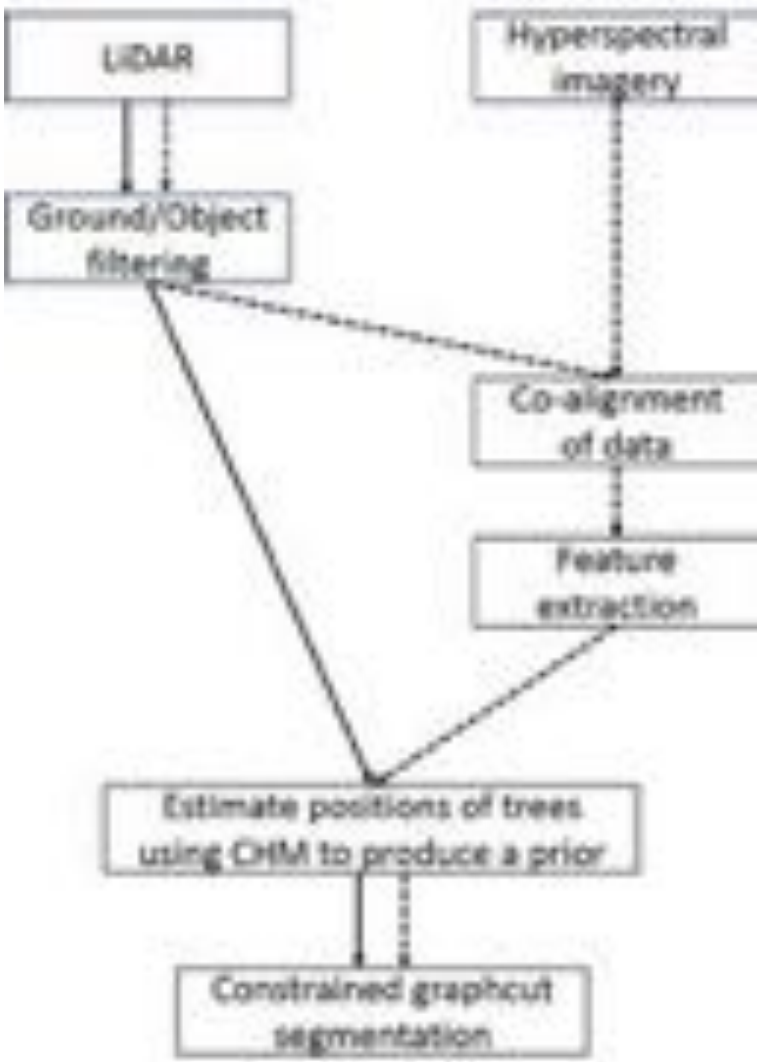
Multi-sensor data: hyperspectral images



J. Lee, CBS, D. Coomes, et al. ’15–

Tree segmentation for forest conservation



J. Lee, CBS, D. Coomes, et al. ’15–

Tree segmentation on the graph

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J. Lee, CBS, D. Coomes, et al. ’15–

Evaluating sexual attractiveness . . .

Joint work with Luca Calatroni, Yves van Gennip and Hannah Rowland





Evaluating sexual attractiveness . . .

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with graph cut!

Bertozzi, Flenner, 2012; L. Calatroni, Y. van Gennip, H. Rowland, CBS, A. Flenner, 2016 forward

Mathematical imaging by examples

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- Arts restoration

Neidthart frescoes



Mathematical analysis can make you fly!

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Mathematics can make you fly! J. Grah, K. Papafitsoros, CBS, EPSRC Science Photo Award '14, Burger, He, CBS '09; CBS, Bertozzi '11; CBS, CUP '15; Chan, Shen '01; Bertalmio et al. '00; Masnou, Morel '98.

Forensics



Low quality fingerprint



Novel enhancement method.

$$\begin{cases} u_t = \operatorname{div}\left(D\left(\mathcal{J}_\rho(\nabla u_\sigma), \mathrm{OF}\right)\nabla u\right) & \text{on } \Omega \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \Omega \\ \langle D\left(\mathcal{J}_\rho(\nabla u_\sigma), \mathrm{OF}\right)\nabla u, \vec{n}\rangle = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where diffusion tensor D is dependent on image structure modelled by tensor $\mathcal{J}_\rho$, and precomputed direction of orientation OF.

Gottschlich, CBS '12; Weickert '98

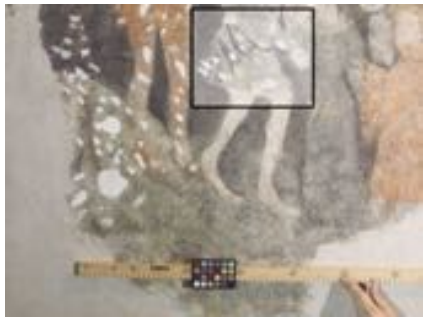
Restoring profanity



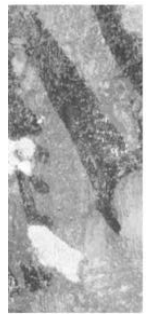
Detail of the Neidhart frescoes (13th century Viennese artwork).

$\Rightarrow$









$$u_t = \alpha\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) + \chi_{\Omega\setminus D}(g-u)$$

TV-H^{-1} image inpainting.

Baatz, Fornasier, Markowich, CBS, '10; Burger, He, CBS '09; CBS, Bertozzi '11; Efros, Leung '99; Bertalmio et al. '03.

Restoring paintings

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Joint work with Marie D'Autume, Spike Bucklow and Stella Panayotova





Restoring paintings



Figure: Adoration of the Shepherds, Sebastiano Del Piombo (1519)

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

Restoring paintings



Figure: A detail

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

Restoring paintings



Figure: Small damages restored with structure (TV) inpainting[1]

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

[1] Chan, Shen '00

Restoring paintings



Figure: Large damages restored with structure & texture inpainting[1]

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

[1] Arias, Facciolo, Caselles, Sapiro '11; CBS, Bertozzi '11; Burger, He, CBS '09; Bertalmio et al. '03; Criminisi et al. '04; Efros, Leung '99.

Unveiling the invisible



Figure: Manuscript by Claude of Franceś Primer

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Unveiling the invisible



Figure: Detail

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Unveiling the invisible



Figure: Infrared image of detail

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Unveiling the invisible



Figure: Unveiled Adam[2]

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

[2] Weickert '13

Unveiling the invisible



Figure: Unveiling the invisible

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Adam and Eve naked again!



Development of methodology, taylor-made for particular application . . . mathematical concepts?

Key mathematical concepts for modern image analysis

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- Large data – small information $\Rightarrow$ sparsity, structured data processing
- Multiple data types – one question $\Rightarrow$ registration, fusion
- Noise in data – robustness $\Rightarrow$ stable processing
- Enormous data sizes to process $\Rightarrow$ scalable and parallelisable optimisation

Thank you very much for your attention!



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